Affine geometry of surfaces and hypersurfaces in $\mathbb{R}^4$

Zusammenfassung
Summary

In the following we will give an overview of our research area. Most of it belongs to the field of affine differential geometry. Geometry, as defined in Felix Klein’s Erlanger Programm, is the theory of invariants with respect to a given transformation group. In this sense affine geometry corresponds to the affine group (general linear transformations and translations) and its subgroups acting on a vector space. So far, most often studied are surfaces in $\mathbb{R}^3$, but also locally strongly convex hypersurfaces in general, mainly with respect to the unimodular (or equiaffine) group. An overview can be found in [LSZ93] and [NS94], see also [LMSS96]. More recently some important global results were achieved (see e.g. [JL05], [TW05]), and the classification of affine hyperspheres with constant sectional curvature is almost complete (see [Vra00] and the references therein).

Most of our research is either about questions concerning surfaces or hypersurfaces in $\mathbb{R}^4$, often in the context of centroaffine geometry.

1 Affine hypersurfaces in $\mathbb{R}^4$ (and $\mathbb{R}^n$)

Motivated by work of Chen [Che93], in [SSVV97] we consider definite centroaffine hypersurfaces $M^n$ and introduce the new intrinsic curvature invariant $\epsilon_M(p) := \frac{n(n-1)}{2} \hat{\kappa}(p) - \sup_{\Pi \in G_2(T_pM)} \hat{K}_p(\Pi)$ of the centroaffine metric $h$, where $\hat{\kappa}$ is the normalized scalar curvature and $\hat{K}$ stands for the sectional curvature. For $n > 2$, we derive an inequality between this invariant and the square norm of the Tchebychev field $T^\sharp$:

$$\epsilon_M \geq -\frac{n^2(n-2)}{2(n-1)} h(T^\sharp, T^\sharp) + \epsilon \frac{1}{2}(n+1)(n-2),$$

where $\epsilon = 1$ if $h$ is positive definite, and $\epsilon = -1$ if $h$ is negative definite. We study the class of hypersurfaces which realize the Chen equality. Since the Tchebychev field vanishes for this class, they are proper affine hyperspheres, centered at the origin. We obtain a wide class of examples and a partial classification. Since this is also important in our later work, we would like to point out, that for the proof the difference tensor $K$ turns out to be crucial. We can find an adapted frame such that $K$ has a special (simple) form. In particular it determines a tangential direction, in which the hypersurface is ruled by arcs of ellipses resp. hyperbolas. Finally, in [SSVV97], we discuss our results under polarization of the hypersurface and relate them to asymptotic spectral results on second order Laplace type operators in affine differential geometry.

In [SV96], we continue our investigation of the new intrinsic curvature invariant for definite centroaffine hypersurfaces. For definite hypersurfaces $f: M^n \rightarrow \mathbb{R}^{n+1}$ the centroaffine normal $\xi = -f$ induces the definite symmetric bilinear form $h$ on $M^n$ (the centroaffine metric).
Whereas in [SSVV97] we concentrated on the positive definite case and only mentioned other results, here we give more details for the negative definite case. The classification turns out to be more complicated than in the positive definite case. We do not only obtain more classes of solutions but also need new methods for the proof.

Not satisfied with the assumption we had to make in addition to the realization of the Chen equality, a new idea arose while we studied the Chen equality for totally real submanifolds $M^3$ in $\mathbb{C}P^3$ (cp. [BSVW01] and [BSV02]). In [KSV01], we use the fact that $SL(4, \mathbb{R})$ is locally isomorphic to $SO(3,3)$. We find that the study of the 2-dimensional submanifolds transverse to the ruling reduces to the problem of investigating 2-dimensional minimal submanifolds of $S^5_3$ whose ellipses of curvature are non-degenerate circles. Here minimal means that the mean curvature vector field vanishes and $S^5_3 \subset \mathbb{R}^6$ is the 5-dimensional (pseudo)sphere of index 3. Unfortunately this only seems to work for elliptic affine hyperspheres. (If the affine metric $h$ is definite we can fix the orientation of the equiaffine normal $\xi(= \pm f)$ such that $h$ is positive definite. Then the sign of the mean curvature $H$ is an invariant, and $M$ is called an elliptic affine hypersphere if $H > 0$ and a hyperbolic affine hypersphere if $H < 0$.)\footnote{In this terminology an ellipsoid is an elliptic and a hyperboloid a hyperbolic affine hypersphere.} We complete the classification of 3-dimensional elliptic affine hyperspheres by a study of 2-dimensional minimal submanifolds of $S^5_3$ whose ellipses of curvature are non-degenerate circles. Hyperbolic affine hyperspheres satisfying the Chen equality are classified in [KV99]. Here, the integrability conditions were solved directly. Again, a crucial point, in both the elliptic and the hyperbolic case, is the existence of an adapted frame such that $K$ has a special (simple) form.

An equiaffine hypersurface is completely determined by its affine metric $h$ and either the induced connection $\nabla$ or the difference tensor $K$. Meanwhile many known classification results prescribe curvature conditions related to $h$ and $\nabla$, there are not that many conditions known for $K$ which allow a classification. Obviously, due to the integrability conditions, conditions imposed on $h$ and $\nabla$ also lead to special properties of $K$ (notice that, by the integrability conditions, $-2h(K(X,Y),Z) = (\nabla_X h)(Y,Z) = C(X,Y,Z)$ defines a totally symmetric $(0,3)$-form, the affine cubic form). Now, the Chen equality prescribes a relation between (the curvatures of) $h$ and $K$, which leads to a special form of $K$ as mentioned before. So what other geometric conditions for $K$ resp. the cubic form (apart of $\nabla^k K = 0$ for some $k \in \mathbb{N}$) might give interesting new classes?\footnote{isotropy group}

The idea to study what we call pointwise symmetries comes from Bryant’s article [Bry05] on special Lagrangian 3-folds. His goal is to classify families of special Lagrangian submanifolds that are characterized by invariant differential geometric conditions, in particular, conditions on the second fundamental form. The second fundamental form can be interpreted as a symmetric cubic form, the fundamental cubic $C_L$. For special Lagrangian submanifolds the cubic form is traceless with respect to the metric (the first fundamental form). So the situation is quite similar to the one in equiaffine geometry (here the apolarity condition gives that the cubic form is traceless with respect to the affine metric). Bryant investigates the question if one can obtain nontrivial families by imposing pointwise conditions on the fundamental cubic. He studies the space of pointwise invariants of a traceless cubic under the special orthogonal group for $n = 3$. It turns out that one has to look at the places where the fundamental cubic has a non-trivial stabilizer, and at the special Lagrangian submanifolds where the stabilizer of the cubic is nontrivial at the generic point. Therefore he classifies the nontrivial $SO(3)$-stabilizers $G$ of traceless cubics in three variables. For each of these he
considers the problem of classifying the special Lagrangian 3-folds whose cubic form at every point has the stabilizer (isomorphic to) $G$ (in our notion: special Lagrangian 3-folds admitting a pointwise $G$-symmetry). One of the resulting families corresponds to the Lagrangian submanifolds satisfying Chen’s equality [BSVW01].

Vrancken transferred this problem to affine geometry in [Vra04]. Studying 3-dimensional positive definite affine hyperspheres, he immediately could start with the classification, since the symmetry groups are these of [Bry05]. Meanwhile no Lagrangian 3-fold admits a pointwise $A_4$-symmetry, positive definite affine 3-spheres with constant sectional curvature do (and only these). If a 3-sphere satisfies the Chen equality, then it admits a pointwise $S_3$-symmetry, but this family is bigger. Maybe the most interesting families are those which admit $\mathbb{Z}_3$- resp. $\text{SO}(2)$-symmetry. They are warped products of either a proper or an improper (i. e. $H \neq 0$ or $H = 0$) positive definite affine 2-sphere resp. quadric and a curve. Finally the condition of $\mathbb{Z}_2$-symmetry is to weak to give an explicit description of this family. Motivated by these results, Vrancken proposed to define pointwise symmetry for affine hypersurfaces in general.

This is done in [SVa] resp. [LS05] (later on the first paper became part of the second one). An affine hypersurface $M$ is said to admit a pointwise $G$-symmetry, if $G$ is a nontrivial (orientation-preserving) subgroup of $\text{Aut}(T_p M)$ for all $p \in M$, which preserves (pointwise) the affine metric $h$, the difference tensor $K$ and the affine shape operator $S$, i. e. for any $X, Y \in T_p M$ and $g \in G$, we have:

$$h(gX_p, gY_p) = h(X_p, Y_p),$$
$$K(gX_p, gY_p) = g(K(X_p, Y_p)),$$
$$S(gX_p) = g(SX_p).$$

Since $S = H^{Id}$ for an affine hypersphere, this gives a generalization of the previous definition. In [LS05] we investigate 3-dimensional positive definite affine hypersurfaces admitting a pointwise symmetry. First we solve the algebraic problem, namely the classification of the $SO(3)$-stabilizers $G$, now of a pair of a traceless symmetric cubic form $C$ and a symmetric quadratic form $S$ in three variables. Since $C(X, Y, Z) = h(K(X, Y, Z)$ and $S(X, Y) = h(S(X, Y))$, this is equivalent to the classification of the non-trivial stabilizers $G$ of the pair $(K, S)$ under the action of $SO(3)$ on an Euclidean vectorspace $(V, h)$. The non-trivial stabilizers are isomorphic to a copy of $SO(3)$, $\mathbb{Z}_2 \times SO(2)$, $SO(2)$, $A_4$, $S_3$, $\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_3$ or $\mathbb{Z}_2$. Compared with the stabilizers of $K$ (cf. [Bry05]) we get in addition $\mathbb{Z}_2 \times SO(2)$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$. Furthermore we fix a representative (canonical form of $K$ and $S$) of each $SO(3)/G$-orbit.

Then, we classify hypersurfaces admitting a pointwise $G$-symmetry for all non-trivial stabilizers $G$ (apart of $\mathbb{Z}_2$). There are no hypersurfaces admitting a pointwise $\mathbb{Z}_2 \times SO(2)$-symmetry. We show that a hypersurface with $G = S_3$ must be a hypersphere, hence the classification can be found in [Vra04]. A hypersurfaces admits a pointwise $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetry if and only if it is affine equivalent to $(x_1 - 1/2 x_3^2)(x_2 - 1/2 x_4^2) = 1$ (and thus it is a locally homogeneous affine hypersurface with rank one shape operator). Finally, we show that for $G = \mathbb{Z}_3$ resp. $G = SO(2)$ we can extend the canonical form of $K$ and $S$ locally and thus obtain information about the coefficients of $K$, $S$ and $\nabla$ from the basic equations of Gauss, Codazzi and Ricci. In particular, it follows that the hypersurface admits a warped product structure $\mathbb{R} \times f N^2$. Then following essentially the same approach as in [Vra04], we classify such hypersurfaces by showing how they can be constructed starting from 2-dimensional positive definite affine spheres resp. quadrics.

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3i. e. open parts of $x_1 x_2 x_3 x_4 = 1$
This last classification can be seen as a generalization of the well known Calabi product of hyperbolic affine spheres and of the constructions for affine spheres considered in [DV94]. The following natural question for a (de)composition theorem, related to the Calabi product and its generalizations in [DV94], gives another motivation for studying 3-dimensional hypersurfaces admitting a pointwise symmetry:

(De)composition Problem. Let $M^n$ be a nondegenerate affine hypersurface in $\mathbb{R}^{n+1}$. Under which conditions do there exist affine hyperspheres $M_1^r$ in $\mathbb{R}^{r+1}$ and $M_2^s$ in $\mathbb{R}^{s+1}$, with $r + s = n - 1$, such that $M = I \times f_1 M_1 \times f_2 M_2$, where $I \subset \mathbb{R}$ and $f_1$ and $f_2$ depend only on $I$ (i.e. $M$ admits a warped product structure)? How can the original immersion be recovered starting from the immersion of the affine spheres?

Of course the first dimension in which the above problem can be considered is three and our study of 3-dimensional affine hypersurfaces with $\mathbb{Z}_3$-symmetry or SO(2)-symmetry provides an answer in that case.

Compared with the (positive) definite case, there is much less known about indefinite affine submanifolds $M^n$, i.e. we have a Pseudo-Riemannian metric on $M^n$. In the classification of affine hyperspheres with constant sectional curvature it turned out to be the less accessible case and the question still remains open for those with vanishing Pick invariant (cp. [Vra00]). Motivated by the interesting families obtained in the foregoing classifications ([Vra04] and [LS05]) and in particular by the composition method, we investigate in [Schb] indefinite affine 3-spheres admitting a pointwise symmetry. We can assume that the affine metric has index two, i.e. the corresponding isometry group is the (special) Lorentz group SO(1,2). Again, we solve the algebraic problem first, namely the classification of the SO(1,2)-stabilizers $G$ of a traceless symmetric cubic form on a Lorentz-Minkowski space $\mathbb{R}^3_1$. The non-trivial stabilizers are isomorphic to a copy of SO(2), SO(1,1), $\mathbb{R}$, $S_3$, $\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_3$ or $\mathbb{Z}_2$. Furthermore we fix a representative (canonical form of $K$) of each SO(1,2)/$G$-orbit.

Then, we classify hyperspheres admitting a pointwise $G$-symmetry for all non-trivial stabilizers $G$ (apart of $\mathbb{Z}_2$). Again we obtain some well-known families: In case of $\mathbb{Z}_2 \times \mathbb{Z}_2$ resp. $\mathbb{R}$, we get the indefinite affine hyperspheres of constant sectional curvature with negative resp. vanishing Pick invariant $J$ (for their classification see [MR92] resp. [DMV00]); those with $J > 0$ are examples for $\mathbb{Z}_2$-symmetry.

Next, we show that indefinite affine 3-spheres admitting a pointwise $\mathbb{Z}_3$-, SO(2)- or SO(1,1)-symmetry are warped product structures $\mathbb{R} \times f N^2$, too. They are warped products of two-dimensional affine spheres ($\mathbb{Z}_3$) or positive definite quadrics (SO(2)) or indefinite quadrics (SO(1,1)), with a curve. Thus we get many new examples of 3-dimensional indefinite affine hyperspheres. Furthermore, we show how one can construct indefinite affine hyperspheres out of two-dimensional quadrics or positive definite affine spheres.

Finally we show that for $G = S_3$ the 3-sphere is locally ruled either by arcs of ellipses (if $H = -1$) resp. hyperbolas (if $H = 1$) or by straight lines (if $H = 0$). Even if we do not have a Riemannian metric (and thus the Chen equality), the situation stays rather similar to the one in [KSV01] resp. [KV99]. We solve the integrability conditions directly, using complex functions. We obtain that $M^3$ is either determined by two functions $h(v,w)$ and $k(v,w)$ satisfying a system of elliptic partial differential equations (generic case) or just by one function $k(v,w)$ satisfying an elliptic partial differential equations (non-generic case).
2 Affine surfaces in $\mathbb{R}^4$

In the thesis [Sch94] we develop a centroaffine theory of (positive) definite oriented surfaces in $\mathbb{R}^4$, using E. Cartan’s method of moving frames. Most of the results presented in [Sch95a], [Sch99] and [Scha] were contents of the thesis, a survey of the results appeared in [Sch95b].

[SV98] is part of an investigation of the centroaffine invariants of surfaces in $\mathbb{R}^4$ which were introduced in [Sch94]. As in every submanifold theory the choice of a transversal plane bundle leads to an induced connection $\nabla$ on the surface. In the equiaffine theory of surfaces in $\mathbb{R}^4$ the assumption that the $\nabla$-geodesics are plane curves is very restrictive. It was shown in [Vra95], that the complex paraboloid is the only definite equiaffine surface with planar $\nabla$-geodesics (if one chooses one of the three equiaffine transversal plane bundles studied by Nomizu and Vrancken in [NV93]). Here we study the same problem in the context of centroaffine geometry.

We prove that a (positive) definite oriented surface with planar $\nabla$-geodesics is centroaffinely equivalent to a surface parametrized by

$$x(s, t) = \frac{1}{\epsilon - \frac{1}{2}(s^2 - t^2)} - st(s, t, s^2 - t^2, 1), \quad h \in \mathbb{R}, \quad \text{and} \quad \epsilon \in \{0, \pm1\}.$$ 

Furthermore we study the properties of this class of surfaces, showing that they are all umbilical and that for all of them $\nabla$ is projectively flat and the normal bundle is flat.

In [MSV95], we study umbilical surfaces in $\mathbb{R}^4$ in the context of equiaffine geometry. A surface is called umbilical, if for each vector belonging to the equiaffine transversal plane bundle introduced by Nomizu and Vrancken in [NV93] the corresponding shape operator is a multiple of the identity. We classify affine umbilical definite surfaces which either have constant curvature or which satisfy $\nabla^\perp g^\perp = 0$. Furthermore, it will be shown that for an affine umbilical definite surface the affine mean curvature vector cannot have constant non-zero length. The results seem to indicate that the condition that $M$ is umbilical seems to be more restrictive for nondegenerate surfaces in $\mathbb{R}^4$ then it is for nondegenerate surfaces in $\mathbb{R}^3$.

As we mentioned before, there are different possibilities to choose a transversal plane for an affine surface in $\mathbb{R}^4$. If we fix on $\mathbb{R}^4$ the standard (torsionfree) connection and a parallel volume form, then there exists an affine invariant metric $g$ on every nondegenerate surface (cf. [NV93]). In this setting we study the different possibilities of choosing a transversal plane bundle in [SV93] and [SVb], stating four principles for the choice of the four unknown functions determining a unique canonical transversal plane. With the restriction to positive definite surfaces, the first three principles lead to a complex one-dimensional family of transversal plane bundles, containing the transversal bundles used in the literature. If we also take into account the fourth principle, one ends up with the transversal bundle introduced in [NV93].

In [SV03] we are working in the more general setting of affine immersions, introduced in [NP87]. Using a method developed in [Sch94] to normalize the second fundamental form allows us to study surfaces with parallel second fundamental form for arbitrary choices of transversal bundles. We classify those affine immersions of a surface in $\mathbb{R}^4$ which are degenerate and have vanishing cubic form (i.e. parallel second fundamental form). It turns out that these are ruled surfaces of three very special types. This completes the classification of parallel surfaces of which the first results were obtained in the beginning of the last century by Blaschke and his collaborators.
References


[SVa] C. Scharlach and L. Vrancken, 3-dimensional affine hypersurfaces admitting a pointwise $SO(2)$- or $\mathbb{Z}_3$-symmetry, Preprint 2003/83, Technische Universität Berlin.


